## Section 2.8 Subspaces of $\mathbb{R}^n$

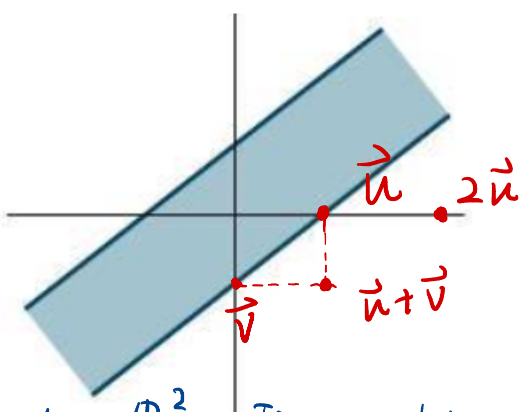
This section focuses on important sets of vectors in $\mathbb{R}^n$ called subspaces.

**Definition: subspace**

A **subspace** of $\mathbb{R}^n$ is any set $H$ in $\mathbb{R}^n$ that has three properties:
a. The zero vector is in $H$.
b. For each $\mathbf{u}$ and $\mathbf{v}$ in $H$, the sum $\mathbf{u}+\mathbf{v}$ is in $H$.
c. For each $\mathbf{u}$ in $H$ and each scalar $c$, the vector $c\mathbf{u}$ is in $H$.

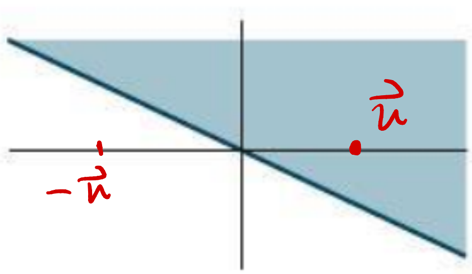
Notice that $\mathbb{R}^n$ itself satisfies the definition. So $\mathbb{R}^n$ is a subspace of itself.

**Example 1.** Assume the sets include the bounding lines. In each case, give a specific reason why the set $H$ is not a subspace of $\mathbb{R}^2$.




$\vec{u}, \vec{v} \in H$
but $\vec{u}+\vec{v} \notin H$.
$2\vec{u} \notin H$.

$H$ is not a subspace for $\mathbb{R}^2$. If we take $\vec{u}, \vec{v}$ in the figure. $\vec{u}, \vec{v}$ are in $H$, but $\vec{u}+\vec{v}$, and $2\vec{u}$ are not in $H$.




If we take $\vec{u} \in H$. as in the figure. then $-\vec{u} \notin H$.

Thus $H$ is not a subspace for $\mathbb{R}^2$.

## Column Space and Null Space of a Matrix

The **column space** of a matrix $A$ is the set $\mathrm{Col}\, A$ of all linear combinations of the columns of $A$.

**Remarks:**

- If $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$, with the columns in $\mathbb{R}^m$, then $\mathrm{Col}\, A$ is the same as $\mathrm{Span}\,\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$.
- Note that $\mathrm{Col}\, A$ equals $\mathbb{R}^m$ only when the columns of $A$ span $\mathbb{R}^m$. Otherwise, $\mathrm{Col}\, A$ is only part of $\mathbb{R}^m$.

If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $\mathrm{Col}\, A = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\} = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} \neq \mathbb{R}^2$

The **null space** of a matrix $A$ is the set $\mathrm{Nul}\, A$ of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Nul $A$ has the properties of a subspace of $\mathbb{R}^n$:

**THEOREM 12** The null space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions of a system $A\mathbf{x} = \mathbf{0}$ of $m$ homogeneous linear equations in $n$ unknowns is a subspace of $\mathbb{R}^n$.

Sketch of proof: a). Since $A\vec{0} = \vec{0}$, $\vec{0} \in \mathrm{Nul}\, A$.

b). If $\vec{u}, \vec{v} \in \mathrm{Nul}A$, then $A\vec{u} = \vec{0}$, $A\vec{v} = \vec{0}$. Thus $A(\vec{u} + \vec{v}) = \vec{0}$. So $\vec{u} + \vec{v} \in \mathrm{Nul}\, A$.

c) If $\vec{u} \in \mathrm{Nul}A$, then $A\vec{u} = \vec{0}$. Thus $A(c\vec{u}) = \vec{0}$. So $c\vec{u} \in \mathrm{Nul}A$.

**Note:** To test whether a given vector $\mathbf{v}$ is in Nul $A$, just compute $A\mathbf{v}$ to see whether $A\mathbf{v}$ is the zero vector.

**Example 2.** Let $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ -6 \\ 3 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 1 \\ 14 \\ -9 \end{bmatrix}$.

(1) Determine if $\mathbf{p}$ is in $\mathrm{Col}\, A$, where $A = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$.

(2) With $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$, determine if $\mathbf{u}$ is in $\mathrm{Nul}\, A$.

ANS: (1) Recall $\mathrm{Col}A$ = the set of all linear combinations of columns of $A$.

So if $\vec{p}$ is in $\mathrm{Col}A \iff \vec{p} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3$ for some $x_1, x_2, x_3$

$\iff A\vec{x} = \vec{p}$ has a solution.

$$[A\ \vec{p}] = \left[\begin{array}{ccc|c} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 6 & 3 & 3 & -9 \end{array}\right] \sim \left[\begin{array}{ccc|c} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 0 & -1 & 3 & -7 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus the augumented matrix corresponds to a consistent system. So $\vec{p}$ is in Col $A$.

(2) To determine if $\vec{u}$ is in Nul $A$, we simply compute $A\vec{u}$. $A\vec{u} = \begin{bmatrix} -3 & -2 & 0 \\ 0 & 2 & -6 \\ 6 & 3 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, i.e. $A\vec{u} = \vec{0}$. So $\vec{u} \in \text{Nul} A$.

## Basis for a Subspace

A **basis for a subspace** $H$ of $\mathbb{R}^n$ is a linearly independent set in $H$ that spans $H$.

### The standard basis for $\mathbb{R}^n$

The columns of an invertible $n \times n$ matrix form a basis for all of $\mathbb{R}^n$ because they are linearly independent and span $\mathbb{R}^n$, by the Invertible Matrix Theorem. One such matrix is the $n \times n$ identity matrix. Its columns are denoted by $\mathbf{e}_1, \ldots, \mathbf{e}_n$ :

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is called the standard basis for $\mathbb{R}^n$. See the following Figure.

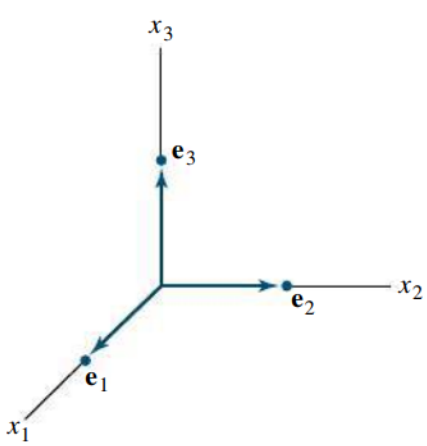



**FIGURE 3**
The standard basis for $\mathbb{R}^3$.

**Theorem 13**

The pivot columns of a matrix $A$ form a basis for the column space of $A$.

Warning: Be careful to use pivot columns of A itself for the basis of $\mathrm{Col}$ A. The columns of an echelon form $B$ are often not in the column space of $A$.

**Example 3.** Given a matrix $A$ and an echelon form of $A$. Find a basis for $\mathrm{Col}\,A$ and a basis for Nul $A$.

$$A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

ANS: Notice that column 1 and column 3 are the pivot columns of A. By Thm 13, a basis for Col A is

$$\left\{ \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} \right\}.$$

Warning: A wrong choice is to select colum 1 and 3 of the echelon form. These columns have zeros in the third entry and could not generate the columns displayed by A.

For Nul A, we first check to solutions for $A\vec{x} = \vec{0}$.

$$\left[\begin{array}{cccc|c} 1 & -3 & 6 & 9 & 0 \\ 0 & 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -3 & 0 & 1.5 & 0 \\ 0 & 0 & 1 & 1.25 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This corresponds to $\begin{cases} x_1 - 3x_2 + 1.5x_4 = 0 \\ x_3 + 1.25x_4 = 0 \\ 0 = 0 \end{cases}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 - 1.5x_4 \\ x_2 \\ -1.25x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1.5 \\ 0 \\ -1.25 \\ 1 \end{bmatrix}$$

This means the solution for $A\vec{x} = \vec{0}$ can be written down as a linear combination of $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1.5 \\ 0 \\ -1.25 \\ 1 \end{bmatrix}$. Moreover, they are linearly independent.

Thus a basis for Nul A is

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1.5 \\ 0 \\ -1.25 \\ 1 \end{bmatrix} \right\}$$

**Exercise 4.** Determine which of the following sets are bases for $\mathbb{R}^2$ or $\mathbb{R}^3$. Justify each answer.

1. $\begin{bmatrix} 3 \\ -8 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -5 \end{bmatrix}$

2. $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$

3. $\begin{bmatrix} 1 \\ -6 \\ -7 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 8 \\ 9 \end{bmatrix}$

**Solution.**

1. No. The vectors cannot be a basis for $\mathbb{R}^3$ because they only span a plane in $\mathbb{R}^3$. Or, point out that the columns of the matrix $\begin{bmatrix} 3 & 6 \\ -8 & 2 \\ 1 & -5 \end{bmatrix}$ cannot possibly span $\mathbb{R}^3$ because the matrix cannot have a pivot in every row. So the columns are not a basis for $\mathbb{R}^3$.
   Note: The *Study Guide* warns students NOT to say that the two vectors here are a basis for $\mathbb{R}^2$.

2. Yes. Place the three vectors into a $3 \times 3$ matrix $A$ and determine whether $A$ is invertible:

$$A = \begin{bmatrix} 1 & -5 & 7 \\ 1 & -1 & 0 \\ -2 & 2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 7 \\ 0 & 4 & -7 \\ 0 & -8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 7 \\ 0 & 4 & -7 \\ 0 & 0 & -5 \end{bmatrix}$$

   The matrix $A$ has three pivots, so $A$ is invertible by the Invertible Matrix Theorem and its columns form a basis for $\mathbb{R}^3$ .

3. No. The vectors are linearly dependent because there are more vectors in the set than entries in each vector. (Theorem 8 in Section 1.7.) So the vectors cannot be a basis for any subspace.